ANNALS OF PHYSICS 79, 542–557 (1973)

How Commutators of Constraints Reflect the Spacetime Structure*

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Received October 18, 1972

The structure constants of the "algebra" of constraints of a parametrized field theory are derived by a simple geometrical argument based exclusively on the path independence of the dynamical evolution; the change in the canonical variables during the evolution from a given initial surface to a given final surface must be independent of the particular sequence of intermediate surface used in the actual evaluation of this change. The requirement of path independence also implies that the theory will propagate consistently only initial data such that the Hamiltonian vanishes. The vanishing of the Hamiltonian arises because the metric of the surface is a canonical variable rather than a c-number. It is not assumed the constraints can be solved to express four of the momenta in terms of the remaining canonical variables. It is shown that the signature of spacetime can be read off from the commutator of two Hamiltonian constraints at different points. The analysis applies equally well irrespective of whether the spacetime is a prescribed Riemannian background or whether it is determined by the theory itself as in general relativity. In the former case the structure of the commutators imposes consistency conditions for a theory in which states are defined on arbitrary spacelike surfaces; whereas, in the later case it provides the conditions for the existence of spacetime—"embeddability" conditions which ensure that the evolution of a three-geometry can be viewed as the "motion" of a three-dimensional cut in a four-dimensional spacetime of hyperbolic signature.

1. INTRODUCTION

Even in flat spacetime it is useful to set up a field theory in such a way that field states are defined on a general spacelike surface, the reason being that the relativistic invariance of the Hamiltonian formalism becomes manifest [1–3]. If, instead of flat spacetime, we work with a nonflat Riemannian background then the use of a curved spacelike hypersurface is unavoidable for the simple reason that three-dimensional planes do not exist in general. Moreover, since general relativity has taught us that spacetime is not flat we may look at the possibility of using flat surfaces as a mere accident peculiar to Minkowski space, and it

* Work supported in part by the National Science Foundation under Grant No. GP-30799X.
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would seem rather unnatural to base a general dynamical theory on such a degenerate case.

When setting the ground for a Schroedinger type quantization on curved surfaces it is natural to look at the Hamilton–Jacobi equation, Hamilton's principal functional S being the classical analog of the quantum wave functional ψ. The observation that in nonrelativistic dynamics the Hamilton–Jacobi equation is a partial differential equation of first order in the dynamical coordinates and also in the time variable led Dirac [4] to the conclusion that we should build a dynamical theory in which the variables describing the surface are treated on the same footing as the dynamical coordinates, and he developed a generalized Hamiltonian formalism based on this idea.

In Dirac's formalism four extra dynamical variables ("surface variables") are introduced in addition to the variables describing the field. In the case of a field theory in Minkowski space these surface variables are most simply taken to be the four Lorentzian coordinates of a generic point on the surface. There is, however, no need to restrict oneself to Lorentzian coordinates; any curvilinear system of coordinates, fixed once and for all, will do the job. The same procedure applies, as well, if the Minkowskian background is replaced by some prescribed Riemannian background.

The introduction of four redundant variables brings into the Hamiltonian formalism four constraints, since the momenta cannot be solved back as functionals of the coordinates and velocities. If the surface variables are denoted by $y^A(x)$ ($A = 0, 1, 2, 3$) and their conjugate momenta by $\pi_A(x)$ (here $x = (x^1, x^2, x^3)$ are curvilinear coordinates on the surface), then these constraints have the form

$$\mathscr{H}_A \equiv \pi_A + K_A = 0, \tag{1}$$

where K_A is *independent of the* π_B but depends in general on the y^A and the field canonical variables. The important point now is that the four quantities $\mathscr{H}_A$ carry all the dynamics of the system at hand. The change in any functional F of the canonical variables (field *and* surface variables), induced by a deformation of the surface $y^A(x) \to y^A(x) + \delta y^A(x)$ is given by the P.B. of F with the Hamiltonian

$$\delta H = \int d^3x \; \delta y^A(x) \, \mathscr{H}_A(x). \tag{2}$$

Also the Hamilton–Jacobi equations are obtained by replacing in (1) π_A by $\delta S/\delta y^A$, with an analogous prescription for the field variables.

A more convenient form of the theory is obtained if the four constraints (1) are projected into one component $\mathscr{H}_\perp$ orthogonal to the surface and three tangential components $\mathscr{H}_r$ by means of the definitions

$$\mathscr{H}_\perp \equiv \mathscr{H}_A n^A; \qquad \mathscr{H}_r = \mathscr{H}_A y^A_{,r}, \tag{3}$$

where n^A is the unit normal to the surface. We then get a system of constraints equivalent to (2), namely

$$\mathscr{H}_\perp \equiv \pi_\perp + K_\perp = 0, \tag{4.a}$$

$$\mathscr{H}_r \equiv \pi_r + K_r = 0. \tag{4.b}$$

When expressed in terms of $\mathscr{H}_\perp$ and $\mathscr{H}_r$ the Hamiltonian (2) takes the form

$$\delta H = \int d^3x\, \delta y^\alpha\, \mathscr{H}_\alpha \equiv \int d^3x\, (\delta y^\perp\, \mathscr{H}_\perp + \delta y^s\, \mathscr{H}_s), \tag{5}$$

where $\delta y^\perp = \epsilon n_A\, \delta y^A (\epsilon = n_B n^B)$ is the normal component of the deformation, and $\delta y^r = g^{rs}\, \delta y_A\, y^A_{,s}$ is the tangential part. The decomposition of a deformation in tangential and orthogonal parts is illustrated in Fig. 1.

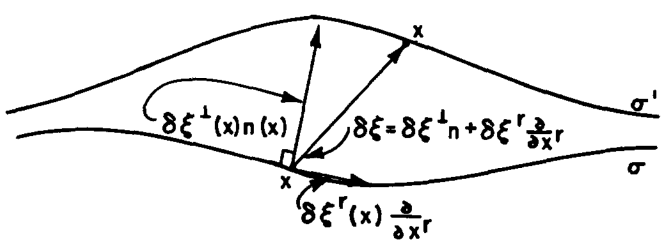

FIG. 1. Deformation of a coordinatized surface. Starting from a given surface σ on which a coordinate system (x) is defined one goes to an infinitesimally neighboring surface σ' by means of a deformation $\delta\xi(x) = \delta\xi^\perp(x)n(x) + \delta\xi^r(x)\ (\partial/\partial x^r)$. Note that the deformation defines σ' not only in the geometrical sense but also sets a coordinate system on σ' by the prescription of giving same coordinates to the points at the tail and at the tip of the deformation. In the terminology of Arnowitt, Deser, and Misner [6], $\delta\xi^\perp$ is δt times the lapse function and $\delta\xi^r$ corresponds to δt times the shift vector.

The advantage of the projected version (3) of the constraints is twofold. First of all, we replace the highly arbitrary description of the motion in terms of the coordinates y^A by a description in terms of deformations of the surface parallel to itself (governed by $\mathscr{H}_r$) and orthogonal to itself (governed by $\mathscr{H}_\perp$) which has an invariant geometrical meaning. Secondly, the change in the field variables under a displacement of the surface parallel to itself amounts only to the response of the field to a change of coordinates in the surface and has no dynamical content, being determined completely by the transformation character of the field. We separate in this way the part of the problem that is trivial from the truly dynamical part which is contained in $\mathscr{H}_\perp$. Moreover, when the constraint $\mathscr{H}_\perp = 0$ is imposed as a restriction on the Hamilton–Jacobi principal functional S in the classical theory or on the state functional ψ in the quantum theory then, thanks to Eq. (6.a), the constraints $\mathscr{H}_r = 0$ follow as a *consequence* of $\mathscr{H}_\perp = 0$, as it has been shown by Moncrief and Teitelboim [5]. This situation is to be contrasted with the formulation based on Eq. (1) in which the two aspects of the problem are mixed and one has to deal with four equations of the same degree of complexity.

The form of $\mathscr{H}_\perp$ and $\mathscr{H}_r$ appearing in the Hamiltonian (5) varies, of course, from theory to theory, but there is one important feature common to all such $\mathscr{H}_\alpha$, namely the fact that the P.B. of any two of them is a linear combination of the $\mathscr{H}_\alpha$ themselves and *this linear combination is the same for all theories.* What we are emphasizing is not the fact that the P.B. of any two constraints is a linear combination of the constraints ("closure")—this merely guarantees the preservation of the constraints during the evolution of the system. What *is* remarkable is that the coefficients in this linear combination ("structure constants") are universal. This fact has been established by Dirac [4] for the case of a parametrized field theory in Minkowski space. *Starting from the form* (1) *of the unprojected constraints* $\mathscr{H}_A$ he has shown that the $\mathscr{H}_\alpha$ obey the following commutation rules

$$[\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')] = (\mathscr{H}^r(x) + \mathscr{H}^r(x'))\, \delta_{,r}(x, x'), \tag{6.a}$$

$$[\mathscr{H}_r(x), \mathscr{H}_\perp(x')] = \mathscr{H}_\perp(x)\, \delta_{,r}(x, x'), \tag{6.b}$$

$$[\mathscr{H}_r(x), \mathscr{H}_s(x')] = (\mathscr{H}_r(x')\, \delta_{,s}(x, x') + \mathscr{H}_s(x)\, \delta_{,r}(x, x')). \tag{6.c}$$

Dirac's procedure suffers from two shortcomings; first of all, the geometrical meaning of the consistency conditions (6) (especially (6.a)) is not clear. Second, his whole procedure depends crucially on the fact that the $\mathscr{H}_\alpha$ are obtained as projections of some $\mathscr{H}_A$'s of the form (1). This is actually satisfied for a theory that was originally "nonparametrized" and in which the surface variables were introduced as canonical variables *a posteriori*. However, there is one very important case in which the theory is "already parametrized," namely, a theory in which the Hamiltonian is given in the form (5) from the very beginning. This is the case of general relativity [6]. Nevertheless, we can take the $\mathscr{H}_\alpha$ of general relativity and compute the P.B. of any two of them. These can be found in a paper by DeWitt [7] and the result is exactly (6). There are two possible reactions to this fact: (a)The coincidence in the P.B's is telling us that there exist some canonical transformation that puts the constraints of general relativity in the form (4) so that Dirac's derivation (generalized to a Riemannian space) applies also to this case. In other words, since Dirac's derivation of (6) is based so strongly on the form (4) of the constraints, we would try to reverse the argument and prove from (6) that the $\mathscr{H}_\alpha$ are of the form (4). If this is the case, it would be a very important result since the main trouble in the canonical quantization of general relativity is that surface variables and field variables are inextricably mixed [8, 9]. The other possible reaction is: (b) Eqs. (6) can be derived without reference to the form (4) of the constraints and, therefore, the fact that the P.B's of the constraints are the same for general relativity and for "deparametrizable" field theories *is not* telling us that general relativity can be deparametrized (i.e., that surface variables and field variables can be separated in a clean way, as in (4)).

We shall derive in this article Eqs. (6) *without any reference whatsoever to a particular form* [*like* (4)] *of the constraints*. The only assumptions will be (i) the constraints are closed (otherwise the theory is inconsistent to start with) and (ii) Hamilton's equations are integrable, that is the change in the canonical variables during the evolution from a given initial surface to a given final surface is independent of the particular sequence of intermediate surfaces used in the actual evaluation of this change. (A consistency requirement termed by Kuchař [10] "path independence of dynamical evolution.")

This way of deriving Eqs. (6) solves the two shortcomings mentioned previously; it provides a clear geometrical interpretation for these P.B. relations, and it shows that alternative (b) is the correct one; the validity of Eqs. (6) for general relativity does not provide evidence for the existence of a canonical transformation that separates dynamical variables from surface variables in gravitation theory. We will return in some detail to the case of general relativity in Section 4; let it be said for the moment that in this case Eq. (6.a) (which is the most interesting of Eqs. (6)) is the condition for the existence of spacetime; it guarantees that the evolution of the dynamical object of the theory (three-dimensional geometry) can be represented as the "motion" of a three-dimensional cut in a four-dimensional manifold of hyperbolic signature.

2. Integrability

We assume that we have a Hamiltonian field theory in which states are defined on an arbitrary spacelike surface. By "surface" we shall understand a surface in the geometrical sense *plus* a system of coordinates (x^1, x^2, x^3) defined on it. The surfaces will be assumed to be embedded in a four-dimensional Riemannian spacetime. The signature of the time direction will be left open and denoted by ϵ (the spacetime metric will then have signature (ϵ, $+$, $+$, $+$)).

In the generalized Hamiltonian formalism of Dirac [4] the change in any functional F of the canonical variables under a deformation δy^α of the surface (recall that Greek indices run over $\alpha = \perp$, 1, 2, 3, with 1, 2, and 3 corresponding to the coordinates x^r on the surface) is given according to (5) by

$$\delta F = \int d^3x \, [F, \delta y^\alpha \mathscr{H}_\alpha], \tag{7}$$

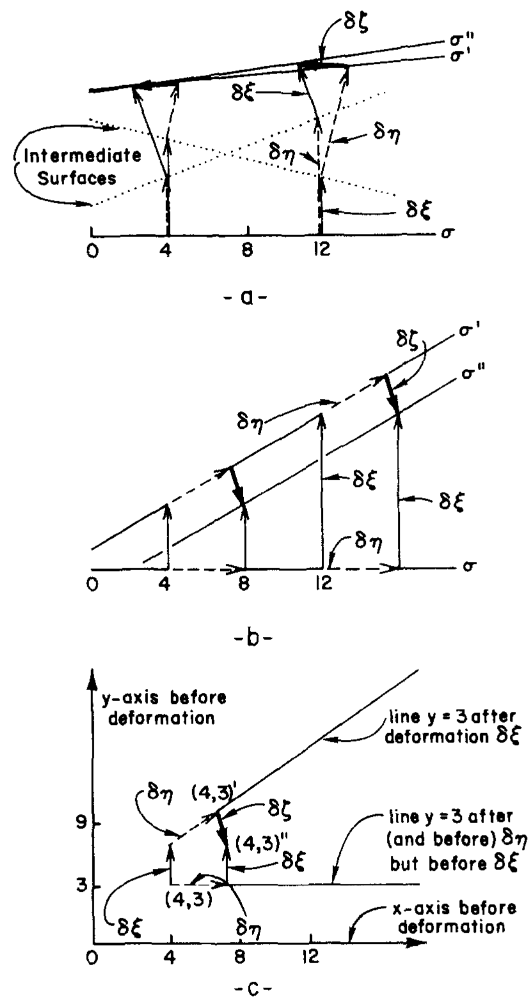

FIG. 2. Deformations do not commute. When two deformations $\delta\xi$ (solid arrow) and $\delta\eta$ (broken arrow) are performed in succession starting from a given surface σ we get different results depending on the order in which the deformations are performed. Recall in this context that "surface" here means "coordinatized surface." That is to say, two surfaces that differ only in the way in which their points are labelled ("active coordinate transformation") are considered to be different. (a) illustrates the noncommutativity of two orthogonal deformations for the simple case of one dimensional "planes" embedded in a two-dimensional Euclidean ($\epsilon = +1$) "spacetime." The coordinate x on the initial surface is proportional to the linear distance measured along the surface. The deformations are given by $\delta\xi = (6 - x/4)n$ and $\delta\eta = \frac{3}{2}(1 + x/4)n$. If $\delta\xi$ is performed first and $\delta\eta$ afterwards we go from σ to σ'. When the order is inverted we go instead to σ''. There is a nonzero vector $\delta\zeta$ (heavy arrow) that deforms σ' onto σ''. In the limit of infinitesimal $\delta\xi$ and $\delta\eta$ the deformation $\delta\zeta$ is tangential to σ' and is given by $\delta\zeta = -(21/8)(\partial/\partial x)$ (Eqs. (25)).

and at the same time the constraints

$$\mathscr{H}_\alpha = 0 \tag{4}$$

hold. Note that on account of (4) the deformation δy^α can be taken outside of the P.B. in (7) even if it depends on the canonical variables without changing δF.

Consider a given surface σ and go to an infinitesimally altered surface σ_1, say, by a deformation $\delta\xi^\alpha$. Go then from σ_1 to another surface σ' by a second deformation $\delta\eta^\alpha$. If the two deformations are performed in reversed order, we will arrive at a final hypersurface σ'' which will be in general different from σ, i.e., normal and tangential deformations are not "holonomic." This feature translates, when expressed by means of the corresponding generators, into a lack of commutativity. The situation is illustrated in Fig. 2. Since σ' and σ'' are in general different, there is a $\delta\zeta$ that deforms σ' into σ''. This "compensating deformation" will have the form

$$\delta\zeta^\gamma(x'') = \int d^3x \int d^3x' \, \kappa^\gamma_{\alpha\beta}(x''; x, x') \, \delta\xi^\alpha(x) \, \delta\eta^\beta(x') + 0((\delta\xi)^2) + 0((\delta\eta)^2), \tag{8}$$

as will be explicitly verified in Section 3.

Note that exchanging $\delta\xi$ and $\delta\eta$ corresponds to exchanging σ' and σ'' which amounts to an inversion of $\delta\zeta$. It follows that

$$\kappa^\gamma_{\alpha\beta}(x''; x, x') = -\kappa^\gamma_{\beta\alpha}(x''; x', x). \tag{9}$$

Now, by repeated use of (7) we find the change

$$F[\sigma'] = F + \int d^3x \, [F, (\delta\xi^\alpha(x) + \delta\eta^\alpha(x)) \, \mathscr{H}_\alpha(x)]$$
$$+ \int d^3x \int d^3x' \, [[F, \delta\eta^\beta(x') \, \mathscr{H}_\beta(x')], \delta\xi^\alpha(x) \, \mathscr{H}_\alpha(x)], \tag{10}$$

(b) illustrates, with the same conventions used as in Fig. (2.a), the noncommutativity of a normal deformation $\delta\xi = (1 + (7/12)x)n$ and a tangential deformation $\delta\eta = 4(\partial/\partial x)$. This time the compensating deformation $\delta\zeta$ is purely normal in the limit of infinitesimal $\delta\xi$ and $\delta\eta$. Equations (25) yield for its value $\delta\zeta = -(7/3)n$. (c) shows that the lack of commutativity of two tangential deformations ("active coordinate transformations") in an Euclidean two dimensional plane (the page). This feature has nothing to do with the embedding of the surface in a higher dimensional space and does not require either the introduction of a metric, in contrast with the situation in Figs. (2.a) and (2.b). A point (x, y) is mapped onto $(x, y)'$ if $\delta\xi$ if performed first and $\delta\eta$ afterwards. If the order is reversed (x, y) is mapped instead onto $(x, y)''$. The compensating deformation $\delta\zeta$ maps $(x, y)'$ onto $(x, y)''$. The deformations considered are $\delta\xi = \frac{2}{3}(x - 1)(\partial/\partial y)$ and $\delta\eta = 3(\partial/\partial x)$, which yields via Eqs. (25), $\delta\zeta = -2(\partial/\partial y)$.

in an arbitrary functional F of the canonical variables (neglecting higher order terms). Here, F and $\mathscr{H}_\alpha$ are evaluated on the original surface σ.

Next we interchange the deformations $\delta\eta^\alpha$ and $\delta\xi^\alpha$, subtract the resulting equation from (10), and use Jacobi's identity to obtain

$$F[\sigma''] - F[\sigma'] = \int d^3x \int d^3x' \, [F, [\delta\xi^\alpha(x) \, \mathscr{H}_\alpha(x), \delta\eta^\beta(x') \, \mathscr{H}_\beta(x')]]. \tag{11}$$

On the other hand, by the very definition of $\delta\zeta$, we have

$$F[\sigma''] - F[\sigma'] = \int d^3x'' \, [F, \delta\zeta^\gamma(x'') \, \mathscr{H}_\gamma(x'')]. \tag{12}$$

The theory will predict a consistent dynamical evolution if and only if (11) and (12) are equal to each other for arbitrary $\delta\xi$ and $\delta\eta$. After taking into account (8) this requirement becomes

$$[F, [\mathscr{H}_\alpha(x), \mathscr{H}_\beta(x')] - \int d^3x'' \, \kappa^\gamma_{\alpha\beta}(x''; x, x') \, \mathscr{H}_\gamma(x'')] = 0. \tag{13}$$

There is a subtlety that has to be considered now. Since the canonical variables are constrained we are entitled, *a priori*, to impose (13) only when the constraint equations (4) hold, i.e., as a weak equation in Dirac's terminology. However, such a requirement is enough to conclude that $[\mathscr{H}_\alpha, \mathscr{H}_\beta] - \int \kappa^\gamma_{\alpha\beta}\mathscr{H}_\gamma$ appearing in (13) vanishes strongly. The argument runs as follows: If $[\mathscr{H}_\alpha, \mathscr{H}_\beta] - \int \kappa^\gamma_{\alpha\beta}\mathscr{H}_\gamma$ depends on at least one canonical variable, we can always find a variable canonically conjugate to this expression. Taking for the arbitrary F in (13) precisely that variable would violate (13) by making its left side a δ-function which is a nonzero "c-number" (i.e., it is independent of the canonical variables)—a quantity that does not vanish, even weakly. We conclude then that $[\mathscr{H}_\alpha, \mathscr{H}_\beta] - \int \kappa^\gamma_{\alpha\beta}\mathscr{H}_\gamma$ is a c-number. Observe now that this c-number vanishes weakly ($[\mathscr{H}_\alpha, \mathscr{H}_\beta]$ has to be a linear combination of the $\mathscr{H}_\gamma$, otherwise the constraints would not be preserved) and that, consequently, it is identically zero, by the very reason that it does not depend on the canonical variables.

We have thus proved that

$$[\mathscr{H}_\alpha(x), \mathscr{H}_\beta(x')] = \int d^3x'' \, \kappa^\gamma_{\alpha\beta}(x''; x, x') \, \mathscr{H}_\gamma(x''), \tag{14}$$

with $\kappa^\gamma_{\alpha\beta}$ defined by (8). Before leaving this point, we remark that the closing relation (14) implies, in turn, that the theory will propagate consistently only such initial data for which the constraints (4) hold. This necessity of the vanishing of the Hamiltonian follows [once (14) is accepted] from the fact that the metric of the surface is a canonical variable rather than a c-number. See the *note added in proof.*

We have seen that the structure constants $\kappa_{\alpha\beta}^{\gamma}$ are determined by a purely geometrical construction, without any assumption about the form of the $\mathscr{H}_\alpha$. We turn now to their explicit evaluation.

3. Evaluation of structure constants

We will determine in this section the "compensating deformation" $\delta\zeta$ as a functional of the elementary deformations $\delta\xi$ and $\delta\eta$ and will then identify the structure constants $\kappa_{\alpha\beta}^{\gamma}$ from Eq. (8).

We have the fundamental relations

$$g^{AB} = \epsilon n^A n^B + g^{rs} y^A_{,r} y^B_{,s}, \tag{15}$$

$$n^A n_A = \epsilon, \tag{16}$$

$$n_A y^A_{,r} = 0, \tag{17}$$

between the spacetime metric g^{AB}, the spacetime coordinates $y^A(x)$ of a point on the surface, the metric on the surface $g^{rs}(x)$ and the unit normal $n^A(x)$.

The components δy^A of a deformation can be expressed in terms of the associated δy^α by means of (15) as

$$\delta y^A = \delta y^\perp n^A + \delta y^r y^A_{,r}. \tag{18}$$

Start from a surface σ defined by $y^A = y^A(x)$ and go to a slightly different surface by means of a deformation $\delta\xi^\alpha$. The equation of this new surface will be

$$y^A = y^A(x) + \delta\xi^\perp n^A + \delta\xi^r y^A_{,r}, \tag{19}$$

and its normal will differ from $n^A(x)$ by a small amount $\delta n^A(x)$ given by

$$\delta n^A(x) = \int d^3x' \frac{\delta n^A(x)}{\delta y^B(x')} (\delta\xi^\perp(x')\, n^B(x') + \delta\xi^r(x')\, y^B_{,r}(x')). \tag{20}$$

This seemingly fancy way of writing δn^A by means of the functional derivative $\delta n^A/\delta y^B$ shows explicitly the dependence of δn^A on $\delta\xi^\alpha$, which is a very useful feature when we replace in some formulas $\delta\xi^\alpha$ by $\delta\eta^\alpha$ and subtract these formulas from each other, as we shall do later. The derivative $\delta n^A/\delta y^B$ can be calculated from (16) and (17), and it is found in the Appendix to be given by

$$\frac{\delta n^A(x)}{\delta y^B(x')} = -\{g^{rs} y^A_{,s} n_B \delta_{,r}(x, x') + g_{CD,B} n^C (g^{AD} - \tfrac{1}{2}\epsilon n^A n^D)\, \delta(x, x')\}, \tag{21}$$

where the coefficient of $\delta_{,r}(x, x')$ is evaluated at the point x.

Go now from the surface defined by (19) to a second surface σ' by a deformation $\delta\eta$. The equation of σ' will be given by an equation similar to (19) with $y^A(x)$ replaced by $y^A(x) + \delta\xi^\perp n^A + \delta\xi^r y^A_{,r}$; $\delta\xi^\alpha$ replaced by $\delta\eta^\alpha$; and n^A replaced by $n^A + \delta n^A$ with δn^A given by (20). This gives

$$\begin{aligned} y^A &= y^A(x) + (\delta\xi^\perp + \delta\eta^\perp)\, n^A + (\delta\xi^r + \delta\eta^r)\, y^A_{,r} \\ &\quad + \delta\eta^\perp \int d^3x' \, \frac{\delta n^A(x)}{\delta y^B(x')} \,(\delta\xi^\perp(x')\, n^B(x') - \delta\xi^r(x')\, y^B_{,r}(x')) \qquad (22) \\ &\quad + \delta\eta^r\, (\delta\xi^\perp\, n^A + \delta\xi^s\, y^A_{,s})_{,r}\, . \end{aligned}$$

If the deformations are performed in opposite order (that is, first $\delta\eta$ and after that $\delta\xi$) we go from the initial surface σ to a final surface σ'', the equation of which has the form (22) with $\delta\xi$ and $\delta\eta$ interchanged. The components $\delta\zeta^A$ of the deformation that transforms σ' into σ'' are, therefore, obtained by subtracting from (22) the same expression with $\delta\xi$ and $\delta\eta$ interchanged. After this is done one needs only to project, in order to get the quantities of interest

$$\delta\zeta^\perp = \epsilon\, \delta\zeta_A\, n^A\, ; \qquad \delta\zeta_r = \delta\zeta_A\, y^A_{,r}\, . \qquad (23)$$

(Actually, for practical purposes it is better to project first and subtract afterwards.)

By making use of the identities

$$n_c n^c_{,r} = -\tfrac{1}{2} g_{CD,B} n^C n^D y^B_{,r}\, , \qquad (24.a)$$

$$n_{B,r}\, y^B_{,s} = n_{B,s}\, y^B_{,r} = -n_B\, y^B_{,sr}\, , \qquad (24.b)$$

which follow from (16) and (17), we get finally

$$\delta\zeta^\perp = -(\delta\eta^r\, \delta\xi^\perp_{,r} - \delta\xi^r\, \delta\eta^\perp_{,r}), \qquad (25.a)$$

$$\delta\zeta^r = \epsilon g^{rs}(\delta\eta^\perp\, \delta\xi^\perp_{,s} - \delta\xi^\perp\, \delta\eta^\perp_{,s}) + (\delta\xi^s\, \delta\eta^r_{,s} - \delta\eta^s\, \delta\xi^r_{,s}). \qquad (25.b)$$

According to (8), we have to write (25) in the form

$$\delta\zeta^\perp(x'') = \int d^3x \int d^3x' \, \kappa^\perp_{\alpha\beta}(x''; x, x')\, \delta\xi^\alpha(x)\, \delta\eta^\beta(x'), \qquad (26.a)$$

$$\delta\zeta^r(x'') = \int d^3x \int d^3x' \, \kappa^r_{\alpha\beta}(x''; x, x')\, \delta\xi^\alpha(x)\, \delta\eta^\beta(x'), \qquad (26.b)$$

in order to identify $\kappa^{\gamma}_{\alpha\beta}$. This yields

$$\kappa^{\perp}_{r\perp}(x''; x, x') = -\kappa^{\perp}_{\perp r}(x''; x', x) = \delta(x'', x)\, \delta_{,r}(x'', x'), \tag{27.a}$$

$$\begin{aligned}\kappa^{r}_{ab}(x''; x, x') &= -\kappa^{r}_{ba}(x''; x', x)\\ &= \delta(x'', x)\, \delta_{,a}(x'', x')\, \delta_b{}^r - \delta(x'', x')\, \delta_{,b}(x'', x)\, \delta_a{}^r,\end{aligned} \tag{27.b}$$

$$\begin{aligned}\kappa^{r}_{\perp\perp}(x''; x, x') &= -\kappa^{r}_{\perp\perp}(x''; x', x)\\ &= \epsilon g^{rs}(x'')(\delta(x'', x')\, \delta_{,s}(x'', x) - \delta(x'', x)\, \delta_{,s}(x'', x')),\end{aligned} \tag{27.c}$$

all others being zero.

Introducing finally expressions (27) into Eq. (14) one gets for the P.B.'s of any pair of constraints,

$$[\mathscr{H}_\perp(x), \mathscr{H}_\perp(x')] = -\epsilon(\mathscr{H}^r(x) + \mathscr{H}^r(x')\, \delta_{,r}(x, x'), \tag{28.a}$$

$$[\mathscr{H}_r(x), \mathscr{H}_\perp(x')] = \mathscr{H}_\perp(x)\, \delta_{,r}(x, x'), \tag{28.b}$$

$$[\mathscr{H}_r(x), \mathscr{H}_s(x')] = (\mathscr{H}_r(x')\, \delta_{,s}(x, x') + \mathscr{H}_s(x)\, \delta_{,r}(x, x')). \tag{28.c}$$

4. Concluding Remarks

We have derived the P.B. relations (28) from a single requirement—integrability of Hamilton's equations. Equations (28.b) and (28.c) are the same as (6.b) and (6.c), and they are not very surprising. As it is well known, (28.c) says that the $\mathscr{H}_r$ are the generators of coordinate changes in the surface, as it should have been expected by observing that setting $\delta y^\perp = 0$ in the Hamiltonian (5) corresponds to a motion of the surface parallel to itself. When the constraint $\mathscr{H}_r = 0$ is imposed as a restriction on the Hamilton–Jacobi functional S in the classical theory or on the wave functional ψ in the quantum theory, the resulting equation says only that S and ψ do not depend upon the parametrization x^r of the surface. Equation (28.b) says that $\mathscr{H}_\perp$ is a scalar density; this comes from the fact that we have written the Hamiltonian (5) as an integral over d^3x instead of as an integral over the proper volume $g^{1/2}d^3x$. Although nothing could prevent us from adopting this last convention, it would be a rather awkward thing to do, because $g^{1/2}$ depends on the canonical variables and, when taking P.B.'s, this would lead to extra terms which would unnecessarily complicate the derivations.

The really interesting equation is (28.a). Note first of all that it involves explicitly the signature ϵ. This means that we can read off immediately the signature of spacetime from this equation. (This point is quite independent of the deeper difficulties in formulating a Hamiltonian formalism in an elliptic spacetime [1]).

This feature is of particular interest in the case of general relativity, because the usual four-dimensional formalism of Einstein's equations does not contain any reference to the signature; whereas, Eq. (28.a) already implies that spacetime will have hyperbolic signature through the fact that the $\mathscr{H}_\perp$ of general relativity satisfies (28.a) with $\epsilon = -1$. (This has been fed into the Hamiltonian formulation by using the Gauss–Codazzi equations with $\epsilon = -1$ when going to a $3 + 1$ splitting.)

When working in a parametrized field theory on a given background, Eq. (28.a) ensures, as it should be clear from the derivation in Sections 2 and 3, that the dynamical evolution is path independent. In the case of general relativity the interpretation is a little more delicate, since there is no background, but the spacetime is determined by the theory itself as a stacking of 3-geometries. At the beginning of Section 3 we have emphasized an assumption which is rather trivial when working on a background but which is not trivial at all in general relativity, namely, the assumption that all the surfaces are embedded in a four-dimensional spacetime. This statement is expressed mathematically by Eq. (28.a) for general relativity, and it is the requirement that is far from being easy to satisfy (in contrast to (28.b) and (28.c)). In fact, Eq. (28.b) restricts rather severely the form of $\mathscr{H}_\perp$, and it should play an important role in any derivation of general relativity based on "first principles." It is appropriate to quote in this context John Wheeler [11]:

> If one did not know the Einstein–Hamilton–Jacobi equation
>
> $$\mathscr{H}_\perp\left[g_{ij}, \frac{\delta S}{\delta g_{ij}}\right] = g^{-1/2}(g_{ik}g_{jl} - \frac{1}{2}g_{ij}g_{kl})\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} + {}^{(3)}R = 0, \qquad (29)$$
>
> how might one hope to derive it straight off from plausible first principles, without ever going through the formulation of the Einstein field equations themselves? ... The central starting point in the proposed derivation would necessarily seem to be "embeddability." ... In what way would one violate the "condition of embeddability" if, for example, one left the differential operator unchanged in (29) but replaced the term ${}^{(3)}R$ by the square or by some other function of the curvature scalar?

Equation (28.a) is the condition for embeddability (with $\epsilon = -1$ for an hyperbolic spacetime).

In order to derive the form of $\mathscr{H}_\perp$ from the embeddability requirement we should write a rather general form for $\mathscr{H}_\perp$ and then impose the relation (28.a). The right side of (28.a) contains only the generators of coordinate changes $\mathscr{H}_r$ and is, therefore, known from the tensorial nature of the variables that come into play. For example, for pure gravitational field we have [4] $\mathscr{H}_r = -2\pi^s_{r|s}$. For the gravitational field in interaction with other fields there will be extra terms in $\mathscr{H}_r$ which are, in each case, uniquely determined. In this more general case of gravitation in interaction with other fields, Eq. (28.a) not only guarantees the embeddability of the 3-geometries in a spacetime but also ensures that these additional fields evolve consistently within this spacetime.

A derivation of the form of the $\mathscr{H}_\perp$ of general relativity from "first principles," starting from the commutation relations (28) has been given by Hojman, Kuchař, and Teitelboim [12].

Note added in proof. We remarked at the end of Section 2 that the closing relation (14) [with $\kappa^{\gamma}_{\alpha\beta}$ given by (27)] implies that the dynamical evolution will be path independent *only when the constraints* (4) *hold.* The argument is rather subtle, and we have found it appropriate to outline it here. A further discussion may be found in Refs. [12] and [14].

The first step is to realize that the change of an arbitrary functional F of the canonical variables under a tangential deformation $\delta\xi^r$ is given by

$$\delta F = \int d^3x[F, \mathscr{H}_r(x)]\, \delta\xi^r(x) \tag{37}$$

even if $\mathscr{H}_r$ is not constrained to be zero. The point we are stressing here is that (37) applies, *with $\delta\xi^r$ outside of the Poisson bracket*, even when the tangential deformation is not a c-number. This follows from the fact that the response of a field quantity to a charge of coordinates on the surface (tangential deformation) depends only on the numerical value of the deformation and is, therefore, independent of any functional dependence on the canonical variables which $\delta\xi^r$ may have.

The second step is to reconsider carefully, in this context, the reasoning leading from (10) to (12). One may assume always (as a particular case) that $\delta\xi^\alpha$ and $\delta\eta^\beta$ in (11) are c-numbers; they can then be safely passed through the P.B.'s even when $\mathscr{H}_\gamma \neq 0$. Eq. (11) can be then rewritten, taking into account (14) as

$$F[\sigma''] - F[\sigma'] = \int d^3x \int d^3x' \int d^3x'' \,[F, \kappa^{\gamma}_{\alpha\beta}(x''; x, x')\, \mathscr{H}_\gamma(x'')]\, \delta\xi^\alpha(x)\, \delta\eta^\beta(x') \tag{38}$$

which, specialized to the case when $\delta\xi$ and $\delta\eta$ are purely normal (the other cases are of no interest, leading to identities) reads [recalling (27)]

$$F[\sigma''] - F(\sigma'] = \int d^3x \int d^3x' \int d^3x'' \,[F, \kappa^{r}_{\perp\perp}(x''; x, x')\mathscr{H}_r(x'')]\, \delta\xi^\perp(x)\, \delta\eta^\perp(x') \tag{39}$$

Observe now what happens to the analog of (12). According to (37), we have to write

$$\begin{aligned} F[\sigma''] - F[\sigma'] &= \int d^3x'' \,[F, \mathscr{H}_r(x'')]\, \delta\xi^r(x'') \\ &\equiv \int d^3x \int d^3x' \int d^3x'' \,[F, \mathscr{H}_r(x'')]\, \kappa^{r}_{\perp\perp}(x''; x')\, \delta\xi^\perp(x)\, \delta\eta^\perp(x') \end{aligned} \tag{40}$$

The evolution will be path-independent if and only if Eqs. (39) and (40) agree, that is when

$$[F, \kappa^{r}_{\perp\perp}(x''; x, x') \mathscr{H}_r(x'')] = [F, \mathscr{H}_r(x'')] \kappa^{r}_{\perp\perp}(x''; x, x'), \tag{41}$$

which is equivalent to

$$[F, \kappa^{r}_{\perp\perp}(x''; x, x')] \mathscr{H}_r(x'') = 0. \tag{42}$$

Equation (42) would be an identity if $\kappa^{r}_{\perp\perp}$ were a c-number. The key point, however, is that this is not so: $\kappa^{r}_{\perp\perp}$ *is not a c-number because it depends on the metric* g_{rs} [Eq. (27.c)]. This is true, both in Dirac's parametrized field theories [Section 1] and in general relativity. Equation (42) can hold for an arbitrary F if and only if

$$\mathscr{H}_r = 0, \tag{43}$$

as can be seen by taking $F = \pi^{rs}$ (conjugate to g_{rs}) in the case of general relativity, or, $F = \pi_A$ (conjugate to the surface variable y^A) in the case of Dirac's parametrized field theories.

The constraints (43) hold on any surface; they must be therefore preserved, in particular, under purely normal deformations. This gives by virtue of (5) and (28.b)

$$\mathscr{H}_\perp = 0. \tag{44}$$

We have thus shown, by exploiting the requirement of path independence, that the closing relations imply the Hamiltonian constraints. The proof depends crucially on the fact that the "structure constant" $\kappa^{r}_{\perp\perp}$ is not actually a "constant" but depends on the field variables. This is precisely what prevents deformations from forming a group, a point discussed in detail in Refs. [12–14].

APPENDIX. CALCULATION OF $\delta n^A(x)/\delta y^B(x')$

The unit normal n^A is defined by the equations

$$g_{AB} n^A n^B = \epsilon, \tag{16}$$

$$g_{AB} n^A y^B_{,r} = 0. \tag{17}$$

Since Eqs. (16) and (17) must be preserved when the surface is varied we get

$$g_{AB,C}\, \delta y^C\, n^A n^B + 2 n_A\, \delta n^A = 0, \tag{29}$$

$$g_{AB,C} n^A y^B_{,r}\, \delta y^C + g_{AB}(\delta n^A\, y^B_{,r} + n^A\, \delta y^B_{,r}) = 0. \tag{30}$$

Now, we can always write (even if δn^A is not a four vector),

$$\delta n^A = \delta n^{\perp} n^A + \delta n^r y^A_{,r}, \tag{31}$$

where

$$\delta n^{\perp} = \epsilon n_A \, \delta n^A, \tag{32}$$

$$\delta n^r = g^{rs} g_{AB} \, \delta n^A \, y^B_{,r} . \tag{33}$$

Inserting (31) into (29) and (30) we get after using (16) and (17) that

$$\delta n^{\perp} = - \tfrac{1}{2} \epsilon g_{AB,C} n^A n^B \, \delta y^C, \tag{34}$$

$$\delta n^r = -g^{rs}(n_A \, \delta y^A_{,r} + g_{AB,C} n^A y^B_{,r} \, \delta y^C), \tag{35}$$

and now we need only to insert back (35) into (31) to get

$$\delta n^D = -g^{rs} n_A y^D_{,s} \, \delta y^A_{,r} - g_{AB,C} \, \delta y^C \, n^A (\tfrac{1}{2} \epsilon n^B n^D + g^{rs} y^B_{,r} y^D_{,s}),$$

which recalling

$$g^{BD} = \epsilon n^B n^D + g^{rs} y^B_{,r} y^D_{,s}, \tag{15}$$

reduces to

$$\delta n^A = -\{ g^{rs} y^A_{,s} n_B \, \delta y^B_{,r} + g_{CD,B} n^C (g^{AD} - \tfrac{1}{2} \epsilon n^A n^D) \, \delta y^B \}, \tag{36}$$

which is equivalent to

$$\frac{\delta n^A(x)}{\delta y^B(x')} = -\{ g^{rs} y^A_{,s} n_B \delta_{,r}(x, x') + g_{CD,B} n^C (g^{AD} - \tfrac{1}{2} \epsilon n^A n^D) \, \delta(x, x') \}. \tag{21}$$

A lucid discussion of the geometrical meaning of the various terms in Eq. (21) has been given by Kuchař [13].

Acknowledgments

I thank Karel Kuchař for very many discussions, invaluable guidance and advice and continuous encouragement. I wish also to express my gratitude to John A. Wheeler for much inspiration and encouragement and for his kind interest in my work. Finally appreciation is extended to the Aspen Center for Physics and the Les Houches Summer School of Theoretical Physics where this work was finished.

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