VIRASORO ALGEBRAS AND COSET SPACE MODELS

P. GODDARD, A. KENT
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

and

D. OLIVE
Blackett Laboratory, Imperial College, London SW7 2BZ, UK

Received 27 November 1984

A previous construction of unitary representations of the Virasoro algebra is extended and interpreted physically in terms of a coset space quark model. The quaternionic projective spaces HP^{n-1} yield the complete range of possible values for the central charge when it is less than unity, namely $1 - 6/(n+1)(n+2)$. The supersymmetric extension is also found.

There has been much interest recently in the role conformal symmetry plays in physical theories, particularly in two dimensions when the conformal algebra is infinite dimensional and comprises two commuting Virasoro algebras $L_n, \bar{L}_n$, each having the relations

$$[L_m, L_n] = (m-n)L_{m+n} + c[\tfrac{1}{12}(m^3 - m)]\delta_{m,-n}, \tag{1}$$

where m and n are integers and c, the central charge of the algebra, is real.

A surprising, yet fruitful, application of this algebra has been in the study of two-dimensional lattice systems [1]. At the critical temperature these systems become scale invariant, and hence by locality conformally invariant. Friedan, Qiu and Shenker (FQS) [2] showed that if the system possesses the "reflection positivity" property then it must furnish a representation of (1) in a positive definite Hilbert space with the unitarity property

$$L_n^+ = L_{-n}, \tag{2}$$

and in which the scale generator $L_0 + \bar{L}_0$ has a spectrum which is bounded below.

It then follows that the state space of the theory decomposes into invariant subspaces with respect to (1), each such subspace possessing a unique highest weight state $|h\rangle$ such that

$$L_0|h\rangle = h|h\rangle, \quad L_m|h\rangle = 0 \quad (m > 0). \tag{3}$$

The numbers $h + \bar{h}$, being the eigenvalues of the scale generator, are the possible critical exponents of the model, and are thus determined by the representation theory of (1).

Given a vector satisfying (3), the corresponding invariant subspace is simply the space built up from it by applying products of L_{-m} $(m > 0)$. The inner product of any two vectors can be calculated from (1), (2) and (3) and it remains to be checked that the Hilbert space so defined is positive definite. FQS showed that this is the case if $c \geqslant 1$ and $h \geqslant 0$, and that the only other values of c and h for which this is possible are given by:

$$c = 1 - 6/(m+1)(m+2) \quad (m \geqslant 1), \tag{4}$$

$$h = h_{p,q}(c) \equiv \frac{[(m+2)p - (m+1)q]^2 - 1}{4(m+1)(m+2)}, \quad (1 \leqslant p \leqslant m,\ 1 \leqslant q \leqslant p). \tag{5}$$

They related these representations for $m = 2, 3, 4$

0370-2693/85/$ 03.30 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

and 5 to the Ising, tricritical Ising, three-state Potts and tricritical three-state Potts models respectively, by comparing the values of h given by (5) with the known critical exponents of these models.

In a previous paper by two of the present authors [3] explicit representations of the algebra (1) with the values for c, h given by $m = 3, 4$ (as well as $m = 2$) in (4) and (5) were presented. This letter follows the previous paper in relying on the mathematical framework of Kac–Moody algebras and the techniques of the operator formalism of dual string theory. We review these briefly, and refer to ref. [3] and references therein for more details.

Suppose that g is the Lie algebra of a compact Lie group, and hence possesses an orthonormal basis in which the structure constants f^{ijk} are totally antisymmetric. The affine Kac–Moody algebra $\hat{g}$ associated with g has generators T^i_n $(1 \leqslant i \leqslant \dim G, n \in \mathbf{Z})$ and commutator

$$[T^i_n, T^j_n] = \mathrm{i} f^{ijk} T^k_{m+n} + k m \delta^{ij} \delta_{m,-n} , \tag{6}$$

where the central element k is real and a positive integral multiple of $\frac{1}{2}\psi^2$ in a highest weight representation. (ψ is the highest root of g.)

A representation of the Virasoro algebra (1) can be obtained from one of $\hat{g}$ as follows. Define the operator

$$\tilde{\mathcal{L}}^{\mathrm{g}}_m = \tfrac{1}{2} \sum_{in} {}^{\circ}_{\circ} T^i_{m+n} T^i_{-n} {}^{\circ}_{\circ} , \tag{7}$$

where

$${}^{\circ}_{\circ} T^i_n T^i_{-n} {}^{\circ}_{\circ} = T^i_{-n} T^i_n \qquad (n > 0) .$$

One can calculate

$$[\tilde{\mathcal{L}}^{\mathrm{g}}_m, T^j_n] = -\beta^{\mathrm{g}} n T^j_{m+n} , \tag{8}$$

where the constant β^{g} is given by

$$\beta^{\mathrm{g}} = k + \tfrac{1}{2} c^{\mathrm{g}}_{\psi} . \tag{9}$$

Here c^{g}_{ψ} is the adjoint representation Casimir operator

$$\sum_{k,l=1}^{\dim \mathrm{g}} f^{ikl} f^{jkl} = c^{\mathrm{g}}_{\psi} \delta^{ij} . \tag{10}$$

It follows from (8) that if one rescales

$$\mathcal{L}^{\mathrm{g}}_m = (1/\beta^{\mathrm{g}}) \tilde{\mathcal{L}}^{\mathrm{g}}_m , \tag{11}$$

then $\mathcal{L}^{\mathrm{g}}_m$ satisfies the Virasoro algebra (1) with

$$c = k d^{\mathrm{g}}_{\psi} / \beta^{\mathrm{g}} = 2 k d^{\mathrm{g}}_{\psi} / (c^{\mathrm{g}}_{\psi} + 2k) , \tag{12}$$

where d^{g}_{ψ} is dim g. This much is familiar.

Now let h be a subalgebra of g, and choose a basis of generators of g such that the first dim h constitute a basis for the generators of h. Then define

$$\tilde{\mathcal{L}}^{\mathrm{h}}_m = \tfrac{1}{2} \sum_{i=1}^{\dim \mathrm{h}} {}^{\circ}_{\circ} T^i_{m+n} T^i_{-n} {}^{\circ}_{\circ} . \tag{13}$$

The preceding argument of eqs. (8)–(12) can be repeated replacing g by h and taking all indices i, j, k, l to run between 1 and dim h.

From (8), (11) and their equivalents for h it follows that

$$[\mathcal{L}^{\mathrm{g}}_m - \mathcal{L}^{\mathrm{h}}_m, T^j_n] = 0 \qquad (1 \leqslant j \leqslant \dim \mathrm{h}) , \tag{14}$$

and hence that

$$[\mathcal{L}^{\mathrm{g}}_m - \mathcal{L}^{\mathrm{h}}_m, \mathcal{L}^{\mathrm{h}}_n] = 0 ,$$

$$[\mathcal{L}^{\mathrm{g}}_m - \mathcal{L}^{\mathrm{h}}_m, \mathcal{L}^{\mathrm{g}}_n - \mathcal{L}^{\mathrm{h}}_n] = [\mathcal{L}^{\mathrm{g}}_m, \mathcal{L}^{\mathrm{g}}_n] - [\mathcal{L}^{\mathrm{h}}_m, \mathcal{L}^{\mathrm{h}}_n] . \tag{15}$$

As $\mathcal{L}^{\mathrm{g}}_m$, $\mathcal{L}^{\mathrm{h}}_m$ each satisfy the Virasoro algebra (1), this shows that so does the difference:

$$K_n = \mathcal{L}^{\mathrm{g}}_n - \mathcal{L}^{\mathrm{h}}_n , \tag{16}$$

with

$$c = 2 k d^{\mathrm{g}}_{\psi} / (c^{\mathrm{g}}_{\psi} + 2k) - 2 k d^{\mathrm{h}}_{\psi} / (c^{\mathrm{h}}_{\psi} + 2k) . \tag{17}$$

where c^{h}_{ψ} is the adjoint representation Casimir operator of h.

$$\sum_{kl=1}^{\dim \mathrm{h}} f^{ikl} f^{jkl} = c^{\mathrm{h}}_{\psi} \delta^{ij} . \tag{18}$$

Note that c given by (17) must be non-negative. If h is not simple, but has two simple factors h_1 and h_2 (so that $\mathrm{h} = \mathrm{h}_1 \oplus \mathrm{h}_2$) eq. (17) is modified to

$$c = 2 k d^{\mathrm{g}}_{\psi} / (c^{\mathrm{g}}_{\psi} + 2k) - 2 k d^{\mathrm{h}_1}_{\psi} / (c^{\mathrm{h}_1}_{\psi} + 2k) - 2 k d^{\mathrm{h}_2}_{\psi} / (c^{\mathrm{h}_2}_{\psi} + 2k) . \tag{19}$$

One further point: if the representation of the T^i_m is on a positive definite Hilbert space, with the unitarity property

$$(T^i_n)^{\dagger} = T^i_{-n} ,$$

then the representations (11) and (16) of (1) will have the unitarity property (2) which we require.

This is relevant to the quaternionic projective space

$$\mathrm{HP}^{n-1} = \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)\times \mathrm{Sp}(1)} . \tag{20}$$

When $\mathrm{g} = \mathrm{sp}(n)$

$$c_\psi^{\mathrm{g}} = \psi^2(n+1) . \tag{21}$$

Hence, if $2k/\psi^2$ takes its smallest possible non-trivial value, namely unity, the central charge (12) for the $\mathrm{g} = \mathrm{sp}(n)$ Virasoro algebra takes the value

$$c = n(2n+1)/(n+2) = 2n - 3 - 6/(n+2) . \tag{22}$$

Thus taking $\mathrm{g} = \mathrm{sp}(n)$, $\mathrm{h}_1 = \mathrm{sp}(n-1)$ and $\mathrm{h}_2 = \mathrm{sp}(1)$ the central charge (19) for the construction (20) yields

$$c = 1 - 6/(n+1)(n+2) , \tag{23}$$

since in the embedding implied by (20) all these symplectic groups have highest roots of the same length. The sequence (23) coincides precisely with the sequence (4) of possible values obtained by FQS. The final step needed to make the construction of the corresponding unitary representations of (1) and (2) concrete is to observe that the Sp(n) affine Kac–Moody algebra (6) with $2k/\psi^2$ equal to unity is realised by the quark model construction of ref. [3], in which the T_n^i are bilinear in Fermi fields (of either Ramond or Neveu–Schwarz type) assigned to the defining representation of Sp(n). This representation is complex (pseudo-real), has $2n$ dimensions and $\kappa_\lambda = \frac{1}{2}\psi^2$ in the notation of ref. [3]. Thus we have constructed a unitary representation of the Virasoro algebra (1) which is quadrilinear in fermion fields, with central charge occurring in the sequence (23).

In (20), we have embodied the quotient G/H of the groups G, H corresponding to our specific choice of g and h. This is because we suspect that our construction (16) can be understood geometrically in terms of the light-cone components of a two-dimensional quark model in which the quarks are assigned to a representation of G but are differentiated with respect to an H-covariant derivative. Since there is no H gauge field kinetic energy, the H gauge field equations of motion simply nullify the h quark currents, leaving the g – h ones. We intend to study this idea further, together with its possible relation to sigma models.

There is an interesting alternative formulation of the construction (20) leading to the sequence (23). The key to the relationship is a generalisation of ideas described in ref. [3]. In the first instance, the "quark model" construction of the affine Kac–Moody algebras $\hat{\mathrm{g}}$ depended upon assigning the "quarks" or Fermi fields to a real representation of g. If we wish to consider a complex representation D we must use $D \oplus D^*$. Then it is possible to construct a u(1) current, commuting with $\hat{\mathrm{g}}$, from the same quarks. If D is peudoreal (i.e. equivalent to its complex conjugate D^* but not real) it has a quaternionic nature, and it is possible to construct a triplet of su(2) currents commuting with the g currents as we now see.

As D is unitary and pseudoreal,

$$D^\dagger D = 1, \quad D^* = \epsilon D \epsilon^{-1}, \quad \det \epsilon \neq 0 . \tag{24}$$

Then $\epsilon^\mathrm{T} = \pm\epsilon$, but since D is not equivalent to a real representation, $\epsilon = -\epsilon^\mathrm{T}$ and hence it and D are of even dimension d_λ. We can take

$$\epsilon = \mathbf{1}_{1/2d_\lambda} \otimes \mathrm{i}\sigma_2 , \tag{25}$$

where $\mathbf{1}_d$ is the $d \times d$ unit matrix and σ_i Pauli matrices. We expand

$$D = \sum_{\alpha=0,1,2,3} a_\alpha \otimes q_\alpha , \tag{26}$$

where a_α is a $((1/2d_\lambda)/2) \times (1/2d_\lambda)/2)$ matrix, $q_0 = \mathbf{1}_2$ and $q_j = \mathrm{i}\sigma_j$ (j = 1, 2, 3) are quaternions. Then pseudoreality implies that the matrices a_α are real. Thus D is "quaternionic".

The representation $D \oplus D^*$ needed for the Kac–Moody construction is found by considering a four-dimensional real representation of the q_i in which they occur as the spinor generators of one su(2) half of an su(2) = su(2) ⊕ su(2) algebra. The generators of the other su(2) furnish the means of constructing the su(2) currents out of the same quarks. We can construct an su(2) Virasoro algebra from the representation of su(2) formed from $1/2d_\lambda$ copies of the real four-dimensional spin 1/2 representation (which we denote by $1/2d_\lambda \cdot [1/2]$) these provide. Since the quarks lie in this representation, the c number is

$$3/2d_\lambda/(1/2d_\lambda + 2) = 3d_\lambda/(d_\lambda + 4) . \tag{27}$$

An illustration is given by taking D equal to the defining, $2n$-dimensional representation of sp(n). Then we have

$$\mathcal{L}^{\mathrm{so}(4n)} = \mathcal{L}^{\mathrm{sp}(n)} + \mathcal{L}^{\mathrm{su}(2)_a} \,, \tag{28}$$

where a stands for $n \cdot [1/2]$, as the c numbers add up across the equation. Hence the energy–momentum tensor constructed out of $\mathrm{sp}(n) \times \mathrm{su}(2)$ quark currents with quarks in the defining representation of $\mathrm{sp}(n)$, equals that of $4n$ free massless real quarks. This generalizes the corresponding results for $\mathrm{so}(n)$ and $\mathrm{u}(n)$ noted in ref. [3]. Thus analogous results hold for all classical simple Lie algebras.

As

$$\mathcal{L}^{\mathrm{so}(4n)} = \mathcal{L}^{\mathrm{so}(4n-4)} + \mathcal{L}^{\mathrm{so}(4)} \,, \tag{29}$$

we have by (28)

$$\mathcal{L}^{\mathrm{sp}(n)} - \mathcal{L}^{\mathrm{sp}(n-1)} - \mathcal{L}^{\mathrm{sp}(1)} = \mathcal{L}^{\mathrm{su}(2)_a} + \mathcal{L}^{\mathrm{su}(2)_b} - \mathcal{L}^{\mathrm{su}(2)_c} \,, \tag{30}$$

where $a = n \cdot [1/2]$ again, $b = (n-1) \cdot [1/2]$ and $c = [1/2]$. The left-hand side is the construction (19), (20) leading to (23), and the right-hand side corresponds to our construction applied to:

$$\mathrm{SU}(2)_b \times \mathrm{SU}(2)_c / \mathrm{SU}(2)_a \,, \tag{31}$$

where the suffices denote the assignment of quark representations.

The left-hand side of (30) provides the nicer geometric picture: we have already mentioned the role it may play in field theories. However, the right-hand side may be more immediately relevant to physics, for the following two reasons. Firstly, because this description involves only spin 1/2 representations of SU(2), we suspect it may be more directly related to spin variables of the corresponding lattice models in statistical physics. Secondly, it is the right-hand side which can be extended to produce all the analogous unitary representations of the supersymmetric extension of the Virasoro algebra, as we explain below.

The original supersymmetry algebra [4] was an extension of (1) in the context of the fermion string model. It is defined by (1) together with:

$$[L_n, G_r] = (\tfrac{1}{2}n - r) G_{n+r} \,, \tag{32}$$

$$[G_r, G_s] = 2L_{r+s} + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})\delta_{r,-s} \,, \tag{33}$$

where $r, s \in \mathbf{Z}$ (Ramond case) or $r, s \in \mathbf{Z} + \frac{1}{2}$ (Neveu–Schwarz case). Highest weight states satisfy

$$G_r |h\rangle = 0 \qquad (r > 0) \,, \tag{34}$$

as well as (3). FQS considered this extended algebra too, and showed that unitary representations either have $c > \frac{3}{2}$ or have

$$c = \tfrac{3}{2}[1 - 8/m(m+2)] \qquad (m \geqslant 2) \,, \tag{35}$$

$$h = h_{p,q}(c) = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \epsilon \,, \qquad (1 \leqslant p \leqslant m, \; 1 \leqslant q \leqslant m+2) \,, \tag{36}$$

where ϵ is 0 and $p - q$ even in the Neveu–Schwarz case, while ϵ is $\frac{1}{16}$ and $p - q$ odd in the Ramond case.

The series (35) of central charges for the unextended algebra (1) can in fact be produced in different ways by natural generalizations of each side of (30). From the left-hand side, the algebra

$$\mathcal{L}^{\mathrm{sp}(m)} - \mathcal{L}^{\mathrm{sp}(m-2)} - \mathcal{L}^{\mathrm{sp}(2)} \,, \tag{37}$$

with all Lie algebras in their defining representations, produces the series (35). (This was pointed out to us by E. Corrigan and D. Fairlie.) However, we have found no way of constructing the extra generators G_r in this case. Indeed, such generators cannot be defined on the entire Ramond or Neveu–Schwarz Fock spaces, since highest weight vectors defined by (36) occur in each, but without respecting the classification given below (36). It is possible, though, that supersymmetry generators might exist on components of these spaces irreducible under the algebra (1).

From the right-hand side, the algebra

$$\mathcal{L}^{\mathrm{su}(2)_b} + \mathcal{L}^{\mathrm{su}(2)_c} - \mathcal{L}^{\mathrm{su}(2)_a} \,, \tag{38}$$

where a is $n \cdot [1/2] + [1]$, b is $n \cdot [1/2]$ and c is $[1]$, also produces the series (35). In this example, the highest weight states do obey the classification given with (36). Indeed they must, as we have constructed generators G_r which obey the relations (32), (33) with L_n given by (37). These are trilinear in the fermion fields, and constructed from tensors invariant under $\mathrm{su}(2)_a$. In the notation of ref. [3], if $\mathrm{su}(2)_b$ and $\mathrm{su}(2)_c$ are represented by the matrices $(M^i)_{\alpha\beta}$ and $(M^i)_{jk}$ respectively (where i, j, k run from 1 to 3; α, β from 1 to $4n$), then the corresponding Kac–Moody generators are given by

$$T_b^i(z) = \tfrac{1}{2}\mathrm{i}M^i_{\alpha\beta} :H^\alpha(z)H^\beta(z): ,$$

$$T_c^i(z) = \tfrac{1}{2}\mathrm{i}M^i_{jk} :H^j(k)H^k(z): . \qquad (39)$$

The supersymmetry generators G_r are the Laurent coefficients of

$$G(z) = [4/(n+2)(n+4)]^{1/2} \times [:H^i(z)T_b^i(z): - \tfrac{1}{6}n :H^i(z)T_c^i(z):] . \qquad (40)$$

For further details, we refer to ref. [5].

We think it is remarkable that there exist such simple unified yet concrete constructions of the Virasoro and super-Virasoro algebra representations corresponding to the discrete spectrum of central charges. We conclude by listing several areas for further research. Although we have not verified that states corresponding to all the highest weights (5) and (36) occur within our representations, we believe this to be the case. The structures of these representations, and their relationship to the physical models whose critical exponents they predict, must be investigated. Clearly, it would also be very interesting to develop the connection between the fermionic and bosonic two-dimensional coset space models.

Adrian Kent is grateful to the UK Science and Engineering Research Council for a studentship.

References

[1] A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333.

[2] D. Friedan, Z. Qiu and S. Shenker, Chicago preprint EFI 83-60, to be published in: Proc. MSRI Workshop on Vertex operators, ed. J. Lepowsky (Springer, Berlin); D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575.

[3] P. Goddard and D. Olive, DAMTP preprint 84/16.

[4] A. Neveu and J.H. Schwarz, Nucl. Phys. B31 (1971) 86; Phys. Rev. D4 (1971) 1109; P. Ramond, Phys. Rev. D3 (1971) 2415.

[5] P. Goddard, A. Kent and D. Olive, in preparation.