Action integrals and partition functions in quantum gravity

G. W. Gibbons* and S. W. Hawking
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England
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One can evaluate the action for a gravitational field on a section of the complexified spacetime which avoids the singularities. In this manner we obtain finite, purely imaginary values for the actions of the Kerr-Newman solutions and de Sitter space. One interpretation of these values is that they give the probabilities for finding such metrics in the vacuum state. Another interpretation is that they give the contribution of that metric to the partition function for a grand canonical ensemble at a certain temperature, angular momentum, and charge. We use this approach to evaluate the entropy of these metrics and find that it is always equal to one quarter the area of the event horizon in fundamental units. This agrees with previous derivations by completely different methods. In the case of a stationary system such as a star with no event horizon, the gravitational field has no entropy.

I. INTRODUCTION

In the path-integral approach to the quantization of gravity one considers expressions of the form

$$Z=\int d[g]\,d[\phi]\exp\{iI[g,\phi]\}\,, \tag{1.1}$$

where $d[g]$ is a measure on the space of metrics g, $d[\phi]$ is a measure on the space of matter fields ϕ, and $I[g,\phi]$ is the action. In this integral one must include not only metrics which can be continuously deformed into the flat-space metric but also homotopically disconnected metrics such as those of black holes; the formation and evaporation of macroscopic black holes gives rise to effects such as baryon nonconservation and entropy production.[1-4] One would therefore expect similar phenomena to occur on the elementary-particle level. However, there is a problem in evaluating the action I for a black-hole metric because of the spacetime singularities that it necessarily contains.[5-7] In this paper we shall show how one can overcome this difficulty by complexifying the metric and evaluating the action on a real four-dimensional section (really a contour) which avoids the singularities. In Sec. II we apply this procedure to evaluating the action for a number of stationary exact solutions of the Einstein equations. For a black hole of mass M, angular momentum J, and charge Q we obtain

$$I=i\pi\kappa^{-1}(M-Q\Phi)\,, \tag{1.2}$$

where

$$\kappa=(r_+-r_-)\,2^{-1}(r_+{}^2+J^2M^{-2})^{-1}\,,$$

$$\Phi=Qr_+(r_+{}^2+J^2M^{-2})^{-1}\,,$$

$$r_\pm=M\pm(M^2-J^2M^{-2}-Q^2)^{1/2}$$

in units such that

$$G=c=\hbar=k=1\,.$$

One interpretation of this result is that it gives a probability, in an appropriate sense, of the occurrence in the vacuum state of a black hole with these parameters. This aspect will be discussed further in another paper. Another interpretation which will be discussed in Sec. III of this paper is that the action gives the contribution of the gravitational field to the logarithm of the partition function for a system at a certain temperature and angular velocity. From the partition function one can calculate the entropy by standard thermodynamic arguments. It turns out that this entropy is zero for stationary gravitational fields such as those of stars which contain no event horizons. However, both for black holes and de Sitter space[8] it turns out that the entropy is equal to one quarter of the area of the event horizon. This is in agreement with results obtained by completely different methods.[1,4,8]

II. THE ACTION

The action for the gravitational field is usually taken to be

$$(16\pi)^{-1}\int R(-g)^{1/2}\,d^4x\,.$$

However, the curvature scalar R contains terms which are linear in second derivatives of the metric. In order to obtain an action which depends only on the first derivatives of the metric, as is required by the path-integral approach, the second derivatives have to be removed by integration by parts. The action for the metric g over a region Y with boundary ∂Y has the form

$$I=(16\pi)^{-1}\int_Y R(-g)^{1/2}d^4x+\int_{\partial Y}B(-h)^{1/2}\,d^3x\,. \tag{2.1}$$

The surface term B is to be chosen so that for metrics g which satisfy the Einstein equations the action I is an extremum under variations of the metric which vanish on the boundary ∂Y but which may have nonzero normal derivatives. This will be satisfied if $B=(8\pi)^{-1}K+C$, where K is the trace of the second fundamental form of the boundary ∂Y in the metric g and C is a term which depends only on the induced metric h, on ∂Y. The term C gives rise to a term in the action which is independent of the metric g. This can be absorbed into the normalization of the measure on the space of all metrics. However, in the case of asymptotically flat metrics, where the boundary ∂Y can be taken to be the product of the time axis with a two-sphere of large radius, it is natural to choose C so that $I=0$ for the flat-space metric η. Then $B=(8\pi)^{-1}[K]$, where $[K]$ is the difference in the trace of the second fundamental form of ∂Y in the metric g and the metric η.

We shall illustrate the procedure for evaluating the action on a nonsingular section of a complexified spacetime by the example of the Schwarzschild solution. This is normally given in the form

$$ds^2=-(1-2Mr^{-1})dt^2+(1-2Mr^{-1})^{-1}dr^2+r^2d\Omega^2. \tag{2.2}$$

This has singularities at $r=0$ and at $r=2M$. As is now well known, the singularity at $r=2M$ can be removed by transforming to Kruskal coordinates in which the metric has the form

$$ds^2=32M^3r^{-1}\exp[-r(2M)^{-1}](-dz^2+dy^2)+r^2d\Omega^2, \tag{2.3}$$

where

$$-z^2+y^2=[r(2M)^{-1}-1]\exp[r(2M)^{-1}], \tag{2.4}$$

$$(y+z)(y-z)^{-1}=\exp[t(2M)^{-1}]. \tag{2.5}$$

The singularity at $r=0$ now lies on the surface $z^2-y^2=1$. It is a curvature singularity and cannot be removed by coordinate changes. However, it can be avoided by defining a new coordinate $\zeta=iz$. The metric now takes the positive-definite or Euclidean form

$$ds^2=32M^3r^{-1}\exp[-r(2M)^{-1}](d\zeta^2+dy^2)+r^2d\Omega^2, \tag{2.6}$$

where r is now defined by

$$\zeta^2+y^2=[r(2M)^{-1}-1]\exp[r(2M)^{-1}]. \tag{2.7}$$

On the section on which ζ and y are real (the Euclidean section), r will be real and greater than or equal to $2M$. Define the imaginary time by $\tau=it$. It follows from Eq. (2.5) that τ is periodic with period $8\pi M$. On the Euclidean section τ has the character of an angular coordinate about the "axis" $r=2M$. Since the Euclidean section is nonsingular we can evaluate the action (2.1) on a region Y of it bounded by the surface $r=r_0$. The boundary ∂Y has topology $S^1\times S^2$ and so is compact.

The scalar curvature R vanishes so the action is given by the surface term

$$I=(8\pi)^{-1}\int[K]d\Sigma. \tag{2.8}$$

But

$$\int K\,d\Sigma=\frac{\partial}{\partial n}\int d\Sigma, \tag{2.9}$$

where $(\partial/\partial n)\int d\Sigma$ is the derivative of the area $\int d\Sigma$ of ∂Y as each point of ∂Y is moved an equal distance along the outward unit normal n. Thus in the Schwarzschild solution

$$\begin{aligned}\int K\,d\Sigma &= -32\pi^2M(1-2Mr^{-1})^{1/2}\\ &\quad\times\frac{d}{dr}[ir^2(1-2Mr^{-1})^{1/2}]\\ &=-32\pi^2iM(2r-3M).\end{aligned} \tag{2.10}$$

The factor $-i$ arises from the $(-h)^{1/2}$ in the surface element $d\Sigma$. For flat space $K=2r^{-1}$. Thus

$$\int Kd\Sigma=-32\pi^2iM(1-2Mr^{-1})^{1/2}\,2r. \tag{2.11}$$

Therefore

$$\begin{aligned}I&=(8\pi)^{-1}\int[K]d\Sigma\\ &=4\pi iM^2+O(M^2r_0^{-1})\\ &=\pi iM\kappa^{-1}+O(M^2r_0^{-1}),\end{aligned} \tag{2.12}$$

where $\kappa=(4M)^{-1}$ is the surface gravity of the Schwarzschild solution.

The procedure is similar for the Reissner-Nordström solution except that now one has to add on the action for the electromagnetic field F_{ab}. This is

$$-(16\pi)^{-1}\int F_{ab}F^{ab}(-g)^{1/2}d^4x. \tag{2.13}$$

For a solution of the Maxwell equations, $F^{ab}{}_{;b}=0$ so the integrand of (2.13) can be written as a divergence

$$F_{ab}F_{cd}g^{ac}g^{bd}=(2F^{ab}A_a)_{;b}. \tag{2.14}$$

Thus the value of the action is

$$-(8\pi)^{-1}\int F^{ab}A_a\,d\Sigma_b. \tag{2.15}$$

The electromagnetic vector potential A_a for the Reissner-Nordström solution is normally taken to be

$$A_a = Qr^{-1}t_{;a}. \tag{2.16}$$

However, this is singular on the horizon as t is not defined there. To obtain a regular potential one has to make a gauge transformation

$$A'_a = (Qr^{-1} - \Phi)t_{;a}, \tag{2.17}$$

where $\Phi = Q(r_+)^{-1}$ is the potential of the horizon of the black hole. The combined gravitational and electromagnetic actions are

$$I = i\pi\kappa^{-1}(M - Q\Phi). \tag{2.18}$$

We have evaluated the action on a section in the complexified spacetime on which the induced metric is real and positive-definite. However, because R, F_{ab}, and K are holomorphic functions on the complexified spacetime except at the singularities, the action integral is really a contour integral and will have the same value on any section of the complexified spacetime which is homologous to the Euclidean section even though the induced metric on this section may be complex. This allows us to extend the procedure to other spacetimes which do not necessarily have a real Euclidean section. A particularly important example of such a metric is that of the Kerr-Newman solution. In this one can introduce Kruskal coordinates y and z and, by setting $\zeta = iz$, one can define a nonsingular section as in the Schwarzschild case. We shall call this the "quasi-Euclidean section." The metric on this section is complex and it is asymptotically flat in a coordinate system rotating with angular velocity Ω, where $\Omega = JM^{-1}(r_+{}^2 + J^2M^{-2})^{-1}$ is the angular velocity of the black hole. The regularity of the metric at the horizon requires that the point (t, r, θ, ϕ) be identified with the point $(t + i2\pi\kappa^{-1}, r, \theta, \phi + i2\pi\Omega\kappa^{-1})$. The rotation does not affect the evaluation of the $\int[K]d\Sigma$ so the action is still given by Eq. (2.18). One can also evaluate the gravitational contribution to the action for a stationary axisymmetric solution containing a black hole surrounded by a perfect fluid rigidly rotating at some different angular velocity. The action is

$$I = i2\pi\kappa^{-1}\left[(16\pi)^{-1}\int_\Sigma R\,K^a d\Sigma_a + 2^{-1}M\right], \tag{2.19}$$

where $K^a\partial/\partial x_a = \partial/\partial t$ is the time-translation Killing vector and Σ is a surface in the quasi-Euclidean section which connects the boundary at $r = r_0$ with the "axis" or bifurcation surface of the horizon $r = r_+$. The total mass, M, can be expressed as

$$M = M_H + 2\int_\Sigma (T_{ab} - \tfrac{1}{2}g_{ab}T)K^a d\Sigma^b, \tag{2.20}$$

where

$$M_H = (4\pi)^{-1}\kappa A + 2\Omega_H J_H. \tag{2.21}$$

M_H is the mass of the black hole, A is the area of the event horizon, and Ω_H and J_H are respectively the angular velocity and angular momentum of the black hole.[9] The energy-momentum tensor of the fluid has the form

$$T_{ab} = (p + \rho)u_a u_b + p g_{ab}, \tag{2.22}$$

where ρ is the energy density and p is the pressure of the fluid. The 4-velocity u_a can be expressed as

$$\lambda u^a = K^a + \Omega_m m^a, \tag{2.23}$$

where Ω_m is the angular velocity of the fluid, m^a is the axial Killing vector, and λ is a normalization factor. Substituting (2.21) and (2.22) in (2.20) one finds that

$$M = (4\pi)^{-1}\kappa A + 2\Omega_H J_H + 2\Omega_m J_m - \int (\rho + 3p)K^a d\Sigma_a, \tag{2.24}$$

where

$$J_m = -\int T_{ab} m^a d\Sigma^b \tag{2.25}$$

is the angular momentum of the fluid. By the field equations, $R = 8\pi(\rho - 3p)$, so this action is

$$I = 2\pi i\kappa^{-1}\left[M - \Omega_H J_H - \Omega_m J_m - \kappa A(8\pi)^{-1} + \int \rho K^a d\Sigma_a\right]. \tag{2.26}$$

One can also apply (2.26) to a situation such as a rotating star where there is no black hole present. In this case the regularity of the metric does not require any particular periodicity of the time coordinate and $2\pi\kappa^{-1}$ can be replaced by an arbitrary periodicity β. The significance of such a periodicity will be discussed in the next section.

We conclude this section by evaluating the action for de Sitter space. This is given by

$$I = (16\pi)^{-1}\int_Y (R - 2\Lambda)(-g)^{1/2}d^4x + (8\pi)^{-1}\int_{\partial Y}[K]d\Sigma, \tag{2.27}$$

where Λ is the cosmological constant. By the field equations $R = 4\Lambda$. If one were to take Y to be the ordinary real de Sitter space, i.e., the section on which the metric was real and Lorentzian, the volume integral in (2.27) would be infinite. However, the complexified de Sitter space contains a section on which the metric is the real positive-definite metric of a 4-sphere of radius $3^{1/2}\Lambda^{-1/2}$. This Euclidean section has no boundary so that the value of this action on it is

$$I=-12\pi i\Lambda^{-1}, \tag{2.28}$$

where the factor of $-i$ comes from the $(-g)^{1/2}$.

III. THE PARTITION FUNCTION

In the path-integral approach to the quantization of a field ϕ one expresses the amplitude to go from a field configuration ϕ_1 at a time t_1 to a field configuration ϕ_2 at time t_2 as

$$\langle \phi_2, t_2 | \phi_1, t_1\rangle = \int d[\phi]\exp(iI[\phi]), \tag{3.1}$$

where the path integral is over all field configurations ϕ which take the values ϕ_1 at time t_1 and ϕ_2 at time t_2. But

$$\langle \phi_2, t_2 | \phi_1, b_1\rangle = \langle \phi_2 | \exp[-iH(t_2-t_1)] | \phi_1\rangle, \tag{3.2}$$

where H is the Hamiltonian. If one sets $t_2-t_1 = -i\beta$ and $\phi_1=\phi_2$ and the sums over all ϕ_1 one obtains

$$\mathrm{Tr}\exp(-\beta H) = \int d[\phi]\exp(iI[\phi]), \tag{3.3}$$

where the path integral is now taken over all fields which are periodic with period β in imaginary time. The left-hand side of (3.3) is just the partition function Z for the canonical ensemble consisting of the field ϕ at temperature $T=\beta^{-1}$. Thus one can express the partition function for the system in terms of a path integral over periodic fields.[10] When there are gauge fields, such as the electromagnetic or gravitational fields, one must include the Faddeev-Popov ghost contributions to the path integral.[11-13]

One can also consider grand canonical ensembles in which one has chemical potentials μ_i associated with conserved quantities C_i. In this case the partition function is

$$Z=\mathrm{Tr}\exp\left[-\beta\left(H-\sum_i \mu_i C_i\right)\right]. \tag{3.4}$$

For example, one could consider a system at a temperature $T=\beta^{-1}$ with a given angular momentum J and electric charge Q. The corresponding chemical potentials are then Ω, the angular velocity, and Φ, the electrostatic potential. The partition function will be given by a path integral over all fields ϕ whose value at the point $(t+i\beta, r, \theta, \phi + i\beta\Omega)$ is exp $(q\beta\Phi)$ times the value at (t, r, θ, ϕ), where q is the charge on the field.

The dominant contribution to the path integral will come from metrics g and matter fields ϕ which are near background fields g_0 and ϕ_0 which have the correct periodicities and which extremize the action, i.e., are solutions of the classical field equations. One can express g and ϕ as

$$g=g_0+\tilde g, \quad \phi=\phi_0+\tilde\phi \tag{3.5}$$

and expand the action in a Taylor series about the background fields

$$I[g,\phi]=I[g_0,\phi_0]+I_2[\tilde g]+I_2[\tilde\phi] + \text{higher-order terms}, \tag{3.6}$$

where $I_2[\tilde g]$ and $I_2[\tilde \phi]$ are quadratic in the fluctuations $\tilde g$ and $\tilde\phi$. If one neglects higher-order terms, the partition function is given by

$$\ln Z = iI[g_0,\phi_0] + \ln\int d[\tilde g]\exp(iI_2[\tilde g]) + \ln\int d[\tilde\phi]\exp(iI_2[\tilde\phi]). \tag{3.7}$$

But the normal thermodynamic argument

$$\ln Z = -WT^{-1}, \tag{3.8}$$

where $W=M-TS-\sum_i \mu_i C_i$ is the "thermodynamic potential" of the system. One can therefore regard $iI[g_0,\phi_0]$ as the contribution of the background to $-WT^{-1}$ and the second and third terms in (3.7) as the contributions arising from thermal gravitons and matter quanta with the appropriate chemical potentials. A method for evaluating these latter terms will be given in another paper.

One can apply the above analysis to the Kerr-Newman solutions because in them the points (t, r, θ, ϕ) and $(t+2\pi i\kappa^{-1}, r, \theta, \phi+2\pi i\Omega\kappa^{-1})$ are identified (the charge q of the graviton and photon are zero). It follows that the temperature T of the background field is $\kappa(2\pi)^{-1}$ and the thermodynamic potential is

$$W=\tfrac{1}{2}(M-\Phi Q), \tag{3.9}$$

but

$$W=M-TS-\Phi Q-\Omega J. \tag{3.10}$$

Therefore

$$\tfrac{1}{2}M=TS+\tfrac{1}{2}\Phi Q+\Omega J, \tag{3.11}$$

but by the generalized Smarr formula[9,14]

$$\tfrac{1}{2}M=\kappa(8\pi)^{-1}A+\tfrac{1}{2}\Phi Q+\Omega J. \tag{3.12}$$

Therefore

$$S=\tfrac{1}{4}A, \tag{3.13}$$

in complete agreement with previous results.

For de Sitter space

$$WT^{-1}=-12\pi\Lambda^{-1}, \tag{3.14}$$

but in this case $W=-TS$, since $M=J=Q=0$ because this space is closed. Therefore

$$S = 12\pi\Lambda^{-1}, \tag{3.15}$$

which again agrees with previous results. Note that the temperature T of de Sitter space cancels out the period. This is what one would expect since the temperature is observer dependent and related to the normalization of the timelike Killing vector.

Finally we consider the case of a rotating star in equilibrium at some temperature T with no event horizons. In this case we must include the contribution from the path integral over the matter fields as it is these which are producing the gravitational field. For matter quanta in thermal equilibrium at a temperature T volume $V \gg T^{-3}$ of flat space the thermodynamic potential is given by

$$WT^{-1} = -i\int p(-\eta)^{1/2}\, d^4x = -pVT^{-1}. \tag{3.16}$$

In situations in which the characteristic wavelengths, T^{-1}, are small compared to the gravitational length scales it is reasonable to use this fluid approximation for the density of thermodynamic potential; thus the matter contributing to the thermodynamic potential will be given by

$$W_m T^{-1} = -i\int p(-g)^{1/2}\, d^4x = T^{-1}\int pK^a\, d\Sigma_a \tag{3.17}$$

(because of the signature of our metric $K^a\, d\Sigma_a$ is negative), but by Eq. (2.26) the gravitational contribution to the total thermodynamic potential is

$$W_g = M - \Omega_m J_m + \int_\Sigma \rho K^a\, d\Sigma_a . \tag{3.18}$$

Therefore the total thermodynamic potential is

$$W = M - \Omega_m J_m + \int_\Sigma (p+\rho) K^a\, d\Sigma_a , \tag{3.19}$$

but

$$p + \rho = \bar{T}s + \sum_i \bar{\mu}_i n_i , \tag{3.20}$$

where $\bar{T}$ is the local temperature, s is the entropy density of the fluid, $\bar{\mu}_i$ is the local chemical potentials, and n_i is the number densities of the ith species of particles making up the fluid. Therefore

$$W = M - \Omega_m J_m + \int_\Sigma \left(\bar{T}s + \sum_i \bar{\mu}_i n_i\right) K^a\, d\Sigma_a . \tag{3.21}$$

In thermal equilibrium

$$\bar{T} = T\lambda^{-1}, \tag{3.22}$$

$$\bar{\mu}_i = \mu_i \lambda^{-1}, \tag{3.23}$$

where T and μ_i are the values of $\bar{T}$ and $\bar{\mu}_i$ at infinity.[9] Thus the entropy is

$$S = -\int s u^a\, d\Sigma_a . \tag{3.24}$$

This is just the entropy of the matter. In the absence of the event horizon the gravitational field has no entropy.

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*Present address: Max-Planck-Institute für Physik und Astrophysik, 8 München 40, Postfach 401212, West Germany. Telephone: 327001.

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