Hamiltonian Reduction of Unconstrained and Constrained Systems

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Recent Letters and Comments discuss the quantization of self-dual two-dimensional Lagrangeans which describe systems that are not explicitly canonical. We make some remarks on the most efficient method for exhibiting the canonical structure.

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It appears that some of our contemporary colleagues are unaware of modern, mathematically based, approaches to quantization, especially of constrained systems. They keep the prejudice that Dirac's method,[1] with its Dirac bracketing and categorization of constraints as first or second class, primary or secondary, "is mandatory." Let us describe an alternative approach. We shall speak of "Hamiltonian formulation," rather than "quantization," because issues of quantum operator ordering are outside our scope.

To begin, one must refrain from viewing a Lagrangean that is first order in time derivatives as necessarily describing a constrained system. Indeed, our starting point is that the dynamical equations of interest be derived from a first-order Lagrangean, which for an initial, simple example we take (summation convention is used)

$$L=p_i\dot{q}^i-H(p,q),\quad i=1,\ldots,n. \tag{1}$$

Upon introducing the $2n$-component phase-space coordinate

$$\xi^i=p_i,\quad i=1,\ldots,n,$$
$$\xi^i=q^i,\quad i=n+1,\ldots,2n,$$

and the Lagrangean one-form $L\,dt$, we can write (1) as

$$L\,dt=\tfrac{1}{2}\,\xi^i f^0_{ij}d\xi^j-V(\xi)dt, \tag{2}$$

where a total time derivative (an exact differential form) has been dropped. Here f^0_{ij} is the symplectic $2n\times 2n$ matrix

$$f^0_{ij}=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}_{ij}. \tag{3}$$

The first term on the right-hand side of (2), $a^0\equiv\frac{1}{2}\,\xi^i f^0_{ij}d\xi^j$, is called the *canonical* one-form; the two-form $f^0\equiv da^0=\frac{1}{2}f^0_{ij}d\xi^i d\xi^j$ is the *symplectic* two-form. In our example a^0 is linear in ξ and f^0 is constant. But we may also consider the more general situation where the symplectic two-form is not constant. The Lagrangean one-form

$$L\,dt=a_i d\xi^i-V(\xi)dt \tag{4}$$

with arbitrary one-form $a\equiv a_i d\xi^i$ gives rise to the Euler-Lagrange equations

$$f_{ij}\dot{\xi}^j=\partial V/\partial\xi^i, \tag{5}$$

$$f_{ij}=\frac{\partial}{\partial\xi^i}a_j-\frac{\partial}{\partial j}a_i. \tag{6}$$

When the two-form $f\equiv da=\frac{1}{2}f_{ij}d\xi^i d\xi^j$ is nonsingular, the matrix inverse to f_{ij} exists, and (5) is equivalent to

$$\dot{\xi}^i=f^{-1}_{ij}\,\partial V/\partial\xi^j. \tag{7}$$

Since the Hamiltonian corresponding to Lagrangean (4) is $V(\xi)$, Eqs. (7) are also Hamiltonian,

$$\dot{\xi}^i=\{V,\xi^i\}=\frac{\partial V}{\partial\xi^j}\{\xi^j,\xi^i\}, \tag{8}$$

provided the basic bracket is defined to be

$$\{\xi^j,\xi^i\}=f^{-1}_{ij}. \tag{9}$$

For the simple case (2), the bracket (9) reduces to the usual $\{p^i,q^j\}=\delta^{ij}$, etc.; the general case for nonsingular two-forms, i.e., for those with an inverse, is no harder. This provides a justification for the commutators posited by Floreanini and Jackiw,[2] as was also briefly indicated in the Appendix to that paper.

It should be stressed that no discussion of constraints need be made: There are none as is clear from (7). Because ξ is already a phase-space variable, it is inappropriate to call $a_i=\partial L/\partial\dot{\xi}^i$ "p^ξ_i, the momentum conjugate to ξ^i," to impose primary, second-class constraints $p^\xi_i-a_i=0$, and to introduce Dirac brackets. Nevertheless, this roundabout procedure does ultimately produce the bracket (9), as Costa and Girotti show.[3]

Constraints arise when f is singular so that the matrix f_{ij} does not possess an inverse. In that case we proceed as follows. It is well known from the theory of differential forms (Darboux's theorem) that for any one-form $a = a_i d\xi^i$, $i=1,\ldots,N$, it is always possible to change variables

$$\xi^i \rightarrow (p^j, q^k z^l), \quad j,k=1,\ldots,n, \quad l=1,\ldots,N-2n, \tag{10}$$

so that a, which defines the canonical variables, takes the standard expression $a = p_i dq^i$ (apart from an additive exact differential—total time derivative). In the unconstrained cases discussed above, with invertible matrix f_{ij}, the variables of type z are absent and (10) effects a "diagonalization" of a nonconstant f_{ij} into the standard form (3), producing the Langrangean (1). In the general case, only a $2n \times 2n$ subblock of f_{ij} is "diagonalized" and $N-2n$ degrees of freedom (corresponding to the z^l) are absent from the new canonical one-form. They survive, however, in the rest of the Lagrangean, which now reads

$$L = p_i dq^i - \Phi(p,q,z)dt. \tag{11}$$

The equations

$$\partial\Phi/\partial z^l = 0 \tag{12}$$

may be used to evaluate the z's in terms of the p's and q's; however, if the matrix $\partial^2\Phi/\partial z^k \partial z^l$ is singular, it will prove impossible to do so. It is easy to convince oneself that in the generic case, after having eliminated as many variables of type z as possible, one is left with an expression linear in the surviving z-type variables. Thus, after diagonalization and z elimination, one arrives at

$$L = p_i \dot{q}^i - H(p,q) - \lambda_l \phi^l(p,q), \tag{13}$$

where now we have renamed the remaining z variables as λ_l—the Lagrange multipliers—*and the* ϕ^l *are the only true constraints in the problem*:

$$\phi^l = 0. \tag{14}$$

Equations (14) may be used to continue the elimination, reducing (13) to the original form (4), $L\,dt = b_i(\eta)d\eta^i - W(\eta)dt$, but with a diminished number of variables. Then the entire procedure must be repeated, until one finally arrives at an unconstrained Lagrangean as in (1). No discussion need be made whether the constraints are first or second class, primary or secondary. It is useful to know that when at a given stage the constraints are first class $[\{\phi^k,\phi^l\}|_{\phi=0}=0]$ and commute with the Hamiltonian $[\{H,\phi^l\}|_{\phi=0}=0]$, or second class $[\det\{\phi^k, \phi^l\}|_{\phi=0}\neq 0]$, then one more step of elimination will lead to the unconstrained Lagrangean. If at some stage the elimination is too difficult to carry out, one may resort to Dirac's approach.[4] Let us stress that the elimination of variables with the help of (12) and (14) is allowed, since our calculations are variational.

Bernstein and Sonnenschein[5] make the interesting point that Siegel's constrained action[6] becomes equivalent to that of Ref. 2 after "path-integral quantization... with first- and second-class constraints." While the assertion is true, it is established by Dirac's machinery. Use of the above formalism renders the argument immediate. Siegel's second-order constrained Lagrangean density

$$\mathcal{L}_S = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \tfrac{1}{2}\lambda(\dot{\phi}-\phi')^2$$

is equivalent to the first-order

$$\mathcal{L} = \pi(\dot{\phi}-\phi') - [2(1+\lambda)]^{-1}(\pi-\phi')^2.$$

[Overdot (prime) denotes time (one-space) differentiation; λ is the Lagrange multiplier.] This is already of the form (13) $[1/2(1+\lambda)\rightarrow\lambda]$, and the solution of the constraint $\pi=\phi'$ leaves $\mathcal{L} = \phi'\dot{\phi} - (\phi')^2$ as in Ref. 2.

For an illustration of the general method when rediagonalization is required, let us show that spinor electrodynamics is quantized without choosing a gauge. The first-order Lagrangean density

$$\mathcal{L} = -\mathbf{E}\cdot\dot{\mathbf{A}} - \tfrac{1}{2}(\mathbf{E}^2+\mathbf{B}^2) + A^0\nabla\cdot\mathbf{E} + i\psi\gamma^\mu(\partial_\mu + iA_\mu)\psi \tag{15a}$$

is of the form (13) with one Lagrange multipler A^0 that enforces $-\nabla^2\phi = \psi^\dagger\psi \equiv \rho$, where $-\nabla\phi \equiv \mathbf{E}_L$, the longitudinal part of $\mathbf{E} \equiv \mathbf{E}_T + \mathbf{E}_L$. The solution of the constraint leaves (apart from total spatial derivatives)

$$\mathcal{L} = -\mathbf{E}_T\cdot\dot{\mathbf{A}}_T + \rho(\nabla/\nabla^2)\cdot\dot{\mathbf{A}}_L + i\psi^\dagger\dot{\psi} - \tfrac{1}{2}(\mathbf{E}_T^2 - \rho\nabla^{-2}\rho + B^2) + \psi^\dagger\alpha\cdot(i\nabla+\mathbf{A})\psi. \tag{15b}$$

The longitudinal vector potential $\mathbf{A}_L$, $\mathbf{A} \equiv \mathbf{A}_T + \mathbf{A}_L$, enters the canonical one-form of (15b) in an uncanonical way. To rediagonalize, only the fermion field need be redefined:

$$\psi \rightarrow \exp[i(\nabla/\nabla^2)\cdot\mathbf{A}_L]\psi.$$

Then we are left with

$$\mathcal{L} = -\mathbf{E}_T\cdot\dot{\mathbf{A}}_T + i\psi^\dagger\dot{\psi} - H_C, \tag{15c}$$

where H_C is the Coulomb-gauge Hamiltonian with only the transverse vector potential. The same analysis can be given for any electromagnetic system, and similarly for gravity theory, but there one stops at a Lagrangean in the form (13), with four first-class constraints.[7]

Finally, let us observe that a first-order Lagrangean (4) may be viewed as the $m\rightarrow 0$ limit of a second-order Lagrangean

$$L_{(2)} = \tfrac{1}{2}m\dot{\xi}^i\dot{\xi}^j + a_i\dot{\xi}^i - V(\xi). \tag{16a}$$

Since the Hamiltonian is

$$H_{(2)}=(2m)^{-1}(p_i^{\xi}-a_i)^2+V(\xi), \qquad (16b)$$

the limit may be taken in phase space only when the constraint $p_i^{\xi}=a_i$ is imposed. This observation is useful to the following end. One may wish to compute commutators of some operators $\mathcal{O}$ in the theory (4), but technically it may be too difficult to do so, because f_{ij}^{-1} cannot be explicitly constructed. On the other hand, similar computation in the theory (16) may be straightforward because of the simpler, expanded symplectic structure. We may now assert that if $\mathcal{O}$ commutes with the constraint $p_i^{\xi}-a_i$ when the latter vanishes, the results in the two theories will be the same. Thus, for example, the recent determination of anomalous Gauss generator commutators in a gauged nonlinear σ model with a quadratic kinetic and first-order Wess-Zumino term[8] also establishes that the same result holds in the theory without the quadratic kinetic term.

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